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Reflections on Free-Piston Stirling Engines, Part 2: Stable Operation

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In this paper a new linearization technique of the dynamic balance equations of a free-piston Stirling machine is developed. It takes account of the nonlinear thermo-fluid-dynamic terms inherent in the machine, while keeping the linearity of the differential dynamic equations. This allows the equations of motion to be solved analytically and, therefore, it allows the algebraic relations linking the various machine parameters, established in a companion paper, to be suitably used. The casing motion is also considered. The following advantages are related to the proposed linearization methodology: 1) It gives the correct interpretation of the machine response to variations in the operating conditions because the considered nonlinear terms have a stabilizing effect that cannot be ignored; 2) it can be used to predict the machine performance not only with more accuracy than conventional linear dynamic analyses that fully neglect the nonlinearities, but in a more exhaustive way, allowing the piston stroke and, therefore, the delivered power to be calculated; and 3) it enables the machine to be designed in the initial stages in such a way as to enhance its inherent stability. To illustrate these features, we have considered, as an example of a free-piston Stirling engine, the well-known Space Power Research Engine.

Nomenclature

A = cross-sectional area for moving element
a = coefficient taking into account the nonlinearities
b = coefficient taking into account the nonlinearities
C = heat capacity rate
D = damping coefficient
$\mathcal{F}$ = force
f = frequency, Hz analytical function
M = moving element mass
p = pressure
Q = heat exchanged
R_l = electric resistive load
r = displacer-piston stroke ratio, X_d/X_p
S = stiffness coefficient
T = temperature
t = time
V = volume
$\dot{V}$ = volumetric flow rate
W = work exchanged
X = moving element stroke
x = moving element displacement
Y = moving element relative stroke
y = moving element relative displacement
Δp = pressure drop
ε = effectiveness
θ = piston-displacer relative phase angle
μ_{dc} = displacer-casing mass ratio, M_d/M_c
μ_{pc} = piston-casing mass ratio, M_p/M_c
σ = displacer-piston relative stroke ratio, Y_d/Y_p
ϕ = piston-displacer phase angle
Ψ = pressure drop per unit of volumetric flow rate
ω = angular frequency, $2\pi f$, rad/s

Subscripts
b = bounce space
c = compression space, casing
d = displacer
dis = dissipated
e = expansion space
gs = gas spring
H = gas spring hysteresis losses
h = heater
hf = outside heating fluid
i = inlet
k = cooler
kf = outside cooling fluid
l = linearized
ld–l = load device–load subsystem
lm = laminar flow in the regenerator
p = piston
r = displacer rod
rg = regenerator
t = thermodynamic
tr = turbulent flow in the heater and cooler
u = useful
w = working gas circuit, working gas

Superscripts
n_D = nonlinear dashpot load exponent
n_S = nonlinear reactive load exponent
$\cdot$ = first-order derivative with respect to time
$\cdot\cdot$ = second-order derivative with respect to time
$-$ = average over a cycle
$\wedge$ = average over half a cycle
$'$ = per unit of moving element mass

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Introduction

IN the linear dynamic analyses of free-piston Stirling engine (FPSE) literature presented and published through 1990, the nonlinear terms in the equations of motion of the moving elements (power piston and displacer) have been fully neglected,[1] with the exception of the analysis developed by Chen and Griffin,[2] which investigated nonlinear loads,[2] and of the analysis developed by the authors in past studies, where the nonlinear pressure losses were considered.[3] Afterward (1993), the effects of nonlinear pressure losses and of a cubic nonlinear term into the load force were analytically investigated by Ulusoy and McCaughan.[4]

In this paper the authors introduce a new linearization technique, based on the concept of linear machine dynamically equivalent to the nonlinear one. According to this concept, the various nonlinear terms appearing in the dynamic equations are represented by dynamically equivalent linear terms, determined in such a way as to give the same contribution to the work over a cycle and to the average pressure. The adoption of such a procedure is justified because in the case of a periodic steady operation, which is the only one of interest in our study concerning machine stability, the influence of the various linear and nonlinear elements is basic mainly over a cycle. This follows directly from the energy conservation law, according to which the engine runs at a periodic steady state only if the thermodynamic work delivered over a cycle by the machine is equal to the work lost over a cycle because of both the thermodynamic and mechanical losses plus the work (useful) delivered to the load device–load (LD–L) subsystem.

The inclusion of the nonlinearities produces more realistic results, but above all, it leads to a better understanding of the nature of the various nonlinear effects and of their stabilizing influence on system operation when the power controls (either internal or external) are inoperable. This is a very important step in the initial stages of the FPSE design.[5]

The linearization procedure can be applied to the dynamic governing equations independently of the thermodynamic model assumed for the working gas circuit, the gas spring, and the bounce space, and it also takes into account the casing motion.

FPSE/LD–L Dynamic Behavior

The equations that describe the dynamic behavior of an FPSE connected to a generic LD–L subsystem may be written, in case of a single-cylinder (beta) engine with displacer sprung to ground, where the casing motion may not be ignored, as follows[6]:

$$M_p \ddot{x}_p = (p_w - p_b)A_p - \Delta p_w A_p/2 - D_{b,H}\dot{y}_p + F_{\text{ld-1}} \quad (1)$$

$$M_d \ddot{x}_d = (p_w - p_{\text{gs}})A_r + \Delta p_w(A_d - A_r/2) - D_{\text{gs},H}\dot{y}_d \quad (2)$$

where y_p and y_d represent, respectively, the relative displacements of the power piston and displacer with reference to the casing motion x_c. In other words

$$y_p = x_p - x_c, \qquad y_d = x_d - x_c \quad (3)$$

The casing motion may be linked to the piston and displacer motions by the following approximate relation[6]:

$$x_c = -\mu_{\text{pc}} x_p - \mu_{\text{dc}} x_d \quad (4)$$

The y_p and y_d relative displacements have been introduced in the dynamic analysis because they allow the linearization of the dynamic equations to be simplified when the casing motions are considered, as will be shown later.

If the machine operating conditions are time independent, then the variables appearing in the equations of motion [Eqs. (1) and (2)] of the moving elements, namely, p_w, p_{gs}, p_b, Δp_w, and $F_{\text{ld-1}}$, do not depend on the time t explicitly, but only implicitly by means of y_p, $\dot{y}_p$ and y_d, $\dot{y}_d$, as already explained in a companion paper.[6] Therefore, the following functional dependences are valid:

$$p_w = f(y_p, y_d), \qquad \Delta p_w = f(\dot{y}_p, \dot{y}_d), \qquad p_{\text{gs}} = f(y_d)$$
$$F_{\text{ld-1}} = f(y_p, \dot{y}_p), \qquad p_b = f(y_p)$$

In particular, it may be noted that the pressures p_w, p_{gs}, and p_b are continuous and infinitely differentiable over the following domain:

$$y_p \in [-Y_p/2, +Y_p/2], \qquad y_d \in [-Y_d/2, +Y_d/2]$$

Because the functional dependences previously listed assume in general a nonlinear form, Eqs. (1) and (2) represent a system of nonlinear differential equations in the unknown functions x_p and x_d. Therefore, the listed variables may be linearized by applying the new dynamic linearization technique described briefly in the Introduction, taking into account the casing motion by means of y_p and y_d, as shown in the following sections.

Linearization of the Working Gas Circuit Pressure

The $p_w = f(y_p, y_d)$ pressure of the working gas, following Urieli and Berchowitz's approach,[7] can be expanded in a MacLaurin series

$$p_w = f(y_p, y_d) = \sum_{m=0}^{\infty} \left\{ \frac{1}{m!} \sum_{k=0}^{m} \left[\left(\frac{\partial^m p_w}{\partial y_p^k \partial y_d^{m-k}} \right)_{0,0} \binom{m}{k} y_p^k y_d^{m-k} \right] \right\} \quad (5)$$

where m is the derivative order. From what was said in the Introduction, it is convenient to linearize the pressure p_w without neglecting the nonlinear terms associated with it, but to represent them by equivalent linear terms. To this aim, because the laws of motion of the moving elements are stationary and periodic functions with zero mean value at a cyclic steady state of the machine, it is important to notice the following:

1) The zero-order term of the Eq. (5) series, $p_w(0, 0)$, does not give any contribution to the work over a cycle, but it does to the average pressure.

2) The first-order term of the Eq. (5) series contributes to the cyclic work, but not to the average pressure:

$$\left[\left(\frac{\partial p_w}{\partial y_p} \right)_{0,0} y_p + \left(\frac{\partial p_w}{\partial y_d} \right)_{0,0} y_d \right]$$

By extrapolating this result to the nonlinear terms of the Eq. (5) series, it is possible to conclude that the nonlinear terms of even-order give a contribution to the average pressure but not to the cyclic work; on the contrary, the nonlinear terms of odd-order give contribution to the cyclic work, but not to the average pressure.

This statement is rigorously valid in the case of sinusoidal laws of motion of the piston and displacer, as can be easily demonstrated.

Therefore, the generic nonlinear term of even-order m can be represented by an equivalent constant term: $a_{w,m} p_w(0, 0)$, giving the same contribution to the average pressure:

$$a_{w,m} p_w(0, 0) \overset{\text{def}}{=} f \oint \left\{ \frac{1}{m!} \sum_{k=0}^{m} \left[\left(\frac{\partial^m p_w}{\partial y_p^k \partial y_d^{m-k}} \right)_{0,0} \binom{m}{k} y_p^k y_d^{m-k} \right] \right\} \mathrm{d}t \quad (6)$$

The generic nonlinear term of odd-order m can be represented, on the contrary, by an equivalent linear term

$$b_{w,m} \left[\left(\frac{\partial p_w}{\partial y_p} \right)_{0,0} y_p + \left(\frac{\partial p_w}{\partial y_d} \right)_{0,0} y_d \right]$$

giving the same contribution to the cyclic work:

$$b_{w,m} \oint \left[\left(\frac{\partial p_w}{\partial y_p} \right)_{0,0} y_p + \left(\frac{\partial p_w}{\partial y_d} \right)_{0,0} y_d \right] \mathrm{d}(V_c + V_e)$$
$$\stackrel{\text{def}}{=} \oint \left\{ \frac{1}{m!} \sum_{k=0}^{n} \left[\binom{m}{k} \left(\frac{\partial^m p_w}{\partial y_p^k \partial y_d^{m-k}} \right)_{0,0} y_p^k y_d^{m-k} \right] \right\} \mathrm{d}(V_c + V_e) \tag{7}$$

The volumes V_e and V_c appearing in the preceding equations are given in Ref. 6. Therefore, the linearized working gas pressure is

$$p_{w,l} = \bar{p}_w + b_w \left[\left(\frac{\partial p_w}{\partial y_p} \right)_{0,0} y_p + \left(\frac{\partial p_w}{\partial y_d} \right)_{0,0} y_d \right] \tag{8}$$

where y_p and y_d are defined by Eq. (3), and

$$\bar{p}_w = a_w p_w(0, 0), \quad a_w = 1 + \sum_{m=2,4}^{\infty} a_{w,m}, \quad b_w = 1 + \sum_{m=3,5}^{\infty} b_{w,m}$$

The coefficients a_w and b_w take into account, respectively, the even- and odd-order nonlinearities associated with the pressure p_w, and are given in Appendix A.

Linearization of the Pressure Drop in the Working Gas Circuit

The pressure drop through the heat exchangers and regenerator of the FPSE working gas circuit may be expressed in the form[6]:

$$\Delta p_w = \Psi_{\mathrm{lm}} \dot{V}_w + \Psi_{\mathrm{tr}} \dot{V}_w |\dot{V}_w| \tag{9}$$

where the terms Ψ_{lm} and Ψ_{tr} (time independent) are defined in Ref. 6, and the volumetric flow rate $\dot{V}_w$ is[6]

$$\dot{V}_w = \tfrac{1}{2} [A_p \dot{y}_p - (2A_d - A_r) \dot{y}_d] \tag{10}$$

Therefore, Δp_w is a quadratic nonlinear function of the $\dot{y}_p$ and $\dot{y}_d$ relative velocities by means of $\dot{V}_w$.

If the nonlinear term of second-order, $\Psi_{\mathrm{tr}} \cdot \dot{V}_w \cdot |\dot{V}_w|$, is neglected in the dynamic linearization procedure, it means not only the turbulent viscous flow energy losses (in the heater and cooler) that it involves are neglected, but most importantly it means its influence on the machine dynamic behavior is neglected, as will be shown later in the text.

Also, in this case it is possible to account for the quadratic nonlinear term by an equivalent linear term: $b_{w,D} \cdot \Psi_{\mathrm{tr}} \cdot \dot{V}_w$, giving the same contribution to the energy dissipated over a cycle

$$\oint (\Psi_{\mathrm{tr}} \dot{V}_w |\dot{V}_w|) \dot{V}_w \, \mathrm{d}t = \oint (b_{w,D} \Psi_{\mathrm{tr}} \dot{V}_w) \dot{V}_w \, \mathrm{d}t \tag{11}$$

For the pressure drop the following linearized expression is valid:

$$\Delta p_{w,l} = (\Psi_{\mathrm{lm}} + b_{w,D} \Psi_{\mathrm{tr}}) \dot{V}_w = \Psi_{w,l} \dot{V}_w \tag{12}$$

where the $b_{w,D}$ coefficient is given in Appendix A, and the $\Psi_{w,l}$ coefficient represents the linearized pressure drop per unit of volumetric flow rate through the heat exchangers and regenerator.

The pressure drop Δp_w appears in the equations of motion [Eqs. (1) and (2)] of the piston and displacer. The force exerted by the linearized pressure drop on the piston motion, bearing in mind the expression of $\dot{V}_w$ given by Eq. (10), can be advantageously written in the following form:

$$-\Delta p_{w,l} A_p / 2 = -\Psi_{w,l} \dot{V}_w A_p / 2$$
$$= -D_{fldp} (\dot{y}_p - \dot{y}_d) + D_{flp} \dot{y}_p \tag{13}$$

The force produced, instead, on the displacer motion by the linearized pressure drop is

$$\Delta p_{w,l} (A_d - A_r/2) = \Psi_{w,l} \dot{V}_w (A_d - A_r/2)$$
$$= -D_{fldp} (\dot{y}_d - \dot{y}_p) - D_{fld} \dot{y}_d \tag{14}$$

The coefficients D_{fldp}, D_{flp}, and D_{fld} (positive) are the damping coefficients as a result of the linearized pressure drop, whose expressions are given in Appendix B. However, the $D_{flp} \dot{y}_p$ term appearing in Eq. (13) actually represents an excitating force rather than a damping one.

Linearization of the Gas Spring and Bounce Space Pressures

Similar to what was discussed for the p_w pressure, the $p_{\mathrm{gs}} = f(y_d)$ pressure exerted by the gas spring on the displacer motion can be expanded in a MacLaurin series:

$$p_{\mathrm{gs}} = f(y_d) = \sum_{m=0}^{\infty} \left[\frac{1}{m!} \left(\frac{\mathrm{d}^m p_{\mathrm{gs}}}{\mathrm{d} y_d^m} \right)_0 y_d^m \right] \tag{15}$$

Because the relative law of motion of the displacer y_d is a periodic and stationary function with zero mean value at a cyclic steady state of the machine and, in addition, the p_{gs} pressure does not give any contribution to the cyclic work,[6] it is possible to conclude the following.

1) The zero-order term of the Eq. (15) series, $p_{\mathrm{gs}}(0)$, and the nonlinear terms of even-order do not contribute to the work exchanged over a cycle, but they do contribute to the average pressure of the gas.

2) The first-order term of the Eq. (15) series, $(\mathrm{d}p_{\mathrm{gs}}/\mathrm{d}y_d)_0 \cdot y_d$, and the nonlinear terms of odd-order do not contribute to the average pressure, nor to the cyclic work.

Thus, the generic nonlinear term of even-order m can be represented by an equivalent constant term: $a_{\mathrm{gs},m} p_{\mathrm{gs}}(0)$, giving the same contribution to the average pressure:

$$a_{\mathrm{gs},m} p_{\mathrm{gs}}(0) \stackrel{\text{def}}{=} f \oint \left[\frac{1}{m!} \left(\frac{\mathrm{d}^m p_{\mathrm{gs}}}{\mathrm{d} y_d^m} \right)_0 y_d^m \right] \mathrm{d}t \tag{16}$$

The generic nonlinear term of odd-order m can be represented, on the contrary, by an equivalent linear term

$$b_{\mathrm{gs},m} \left(\frac{\mathrm{d}p_{\mathrm{gs}}}{\mathrm{d}y_d} \right)_0 y_d$$

giving the same contribution to the work exchanged over a quarter of cycle, i.e., over a half of relative displacer stroke (as listed, the contribution to the cyclic work is zero and would lead, therefore, to an indetermination in the calculation of $b_{\mathrm{gs},m}$):

$$b_{\mathrm{gs},m} \int_0^{Y_d/2} \left[\left(\frac{\mathrm{d}p_{\mathrm{gs}}}{\mathrm{d}y_d} \right)_0 y_d \right] \mathrm{d}V_{\mathrm{gs}} \stackrel{\text{def}}{=} \int_0^{Y_d/2} \left[\frac{1}{m!} \left(\frac{\mathrm{d}^m p_{\mathrm{gs}}}{\mathrm{d}y_d^m} \right)_0 y_d^m \right] \mathrm{d}V_{\mathrm{gs}} \tag{17}$$

where $dV_{gs} = -A_r\, dy_d$.[6] In this way it is possible to account for the nonlinear terms of odd-order m. Thus, the linearized gas spring pressure is

$$p_{gs,l} = \bar{p}_{gs} + b_{gs}\left(\frac{dp_{gs}}{dy_d}\right)_0 y_d$$

where y_d is given by Eq. (3), and

$$\bar{p}_{gs} = a_{gs}p_{gs}(0), \quad a_{gs} = 1 + \sum_{m=2,4}^{\infty} a_{gs,m}, \quad b_{gs} = 1 + \sum_{m=3,5}^{\infty} b_{gs,m}$$

The coefficients a_{gs} and b_{gs} take into account, respectively, the even- and odd-order nonlinearities associated with the pressure p_{gs}, and are given in Appendix A. Similarly, the linearized bounce space pressure is

$$p_{b,l} = \bar{p}_b + b_b\left(\frac{dp_b}{dy_p}\right)_0 y_p$$

where y_p is given by Eq. (3), and

$$\bar{p}_b = a_b p_b(0), \quad a_b = 1 + \sum_{m=2,4}^{\infty} a_{b,m}, \quad b_b = 1 + \sum_{n=3,5}^{\infty} b_{b,m}$$

The coefficients a_b and b_b are also given in Appendix A.

Linearization of the Force Exerted by the LD–L Subsystem

The force resisting the piston motion, F_{ld-1}, appearing in Eq. (1), can be expressed in a quite general form by means of a nonlinear dependence on piston relative displacement y_p and relative velocity $\dot{y}_p$

$$F_{ld-1} = -S_{ld-1}y_p|y_p|^{n_S-1} - D_{ld-1}\dot{y}_p|\dot{y}_p|^{n_D-1} \tag{18}$$

where n_S and n_D are real numbers greater than 1. Every value of the load (L) corresponds to a value for S_{ld-1} and D_{ld-1}. Obviously, the functional dependence of S_{ld-1} and D_{ld-1} on the load is unknown, because a generic LD–L subsystem connected to the engine is being considered. Therefore, the particular form of S_{ld-1} and D_{ld-1}, as well as the values of the exponents n_S and n_D, have to be determined as a function of the adopted load device. In the case of a machine driving a linear alternator–static load (LA–sL) subsystem, their expressions are known and were found by Benvenuto and de Monte as functions of the electric resistive load R_l.[8,9]

Because the force expressed by Eq. (18) has a nonlinear form, it may be conveniently linearized by the usual method. Thus, it may be represented by an equivalent linear force:

$$F_{ld-1,l} = -(b_{ld-,S}S_{ld-1})y_p - (b_{ld-1,D}D_{ld-1})\dot{y}_p \tag{19}$$

giving the same contribution to the cyclic work:

$$\oint y_p|y_p|^{n_S-1}\, dy_p \overset{\text{def}}{=} \oint b_{ld-1,S}y_p\, dy_p \tag{20}$$

$$\oint \dot{y}_p|\dot{y}_p|^{n_D-1}\, dy_p \overset{\text{def}}{=} \oint b_{ld-1,D}\dot{y}_p\, dy_p \tag{21}$$

Equation (20) does not allow the coefficient $b_{ld-1,S}$ to be obtained. In fact, both the nonlinear stiffness force and the equivalent linear stiffness force provide a zero cyclic work and, therefore, Eq. (20) is satisfied for any value of $b_{ld-1,S}$. To avoid this indetermination, it is necessary to require that the work done over a quarter of cycle, i.e., over a half of relative piston stroke, is equal

$$\int_0^{Y_p/2} y_p^{n_S}\, dy_p \overset{\text{def}}{=} \int_0^{Y_p/2} b_{ld-1,S}y_p\, dy_p \tag{22}$$

Equation (21), instead, allows the $b_{ld-1,D}$ coefficient to be calculated readily. The coefficients $b_{ld-1,S}$ and $b_{ld-1,D}$ are given in Appendix A.

Closed-Form Solution of the Linearized Dynamic Balance Equations and FPSE/LD–L Performance

Substituting the linearized expressions of the functions p_w, p_{gs}, p_b, Δp_w, and F_{ld-1}, in Eqs. (1) and (2), and bearing in mind Eqs. (3) and (4), we get the linearized dynamic equations of the machine[6]

$$\ddot{x}_p + D'_{pp}\dot{x}_p + D'_{pd}\dot{x}_d + S'_{pp}x_p + S'_{pd}x_d = 0 \tag{23}$$

$$\ddot{x}_d + D'_{dp}\dot{x}_p + D'_{dd}\dot{x}_d + S'_{dp}x_p + S'_{dd}x_d = 0 \tag{24}$$

where the right-hand terms are equal to zero because the average pressures $\bar{p}_w$, $\bar{p}_{gs}$, and $\bar{p}_b$, are equal in value.[6] The S'_{ij} and D'_{ij} ($i = p, d$; $j = p, d$) coefficients represent the stiffness and damping coefficients per unit of moving element mass of the FPSE/LD–L system and are defined in Appendix B.

Because Eqs. (23) and (24) are linear differential equations, they can be solved analytically.[6] In particular, the solution of the polynomial characteristic equation associated with Eqs. (23) and (24) yields the conditions for the periodic steady operation of the FPSE/LD–L system, characterized by sinusoidal laws of motion of the piston and displacer[6]:

$$x_p(t) = (X_p/2)\sin(\omega t - \phi) \tag{25}$$

$$x_d(t) = (rX_p/2)\sin \omega t \tag{26}$$

In addition, further useful algebraic expressions linking the various machine parameters may be derived. All of these algebraic equations are listed next[6]:

$$\alpha, \beta, \gamma, \delta > 0 \tag{27}$$

$$\beta - \gamma/\alpha = \delta\alpha/\gamma \tag{28}$$

$$\omega^2 = \gamma/\alpha \tag{29}$$

$$r = \left(\frac{C^2 + D^2}{A^2 + B^2}\right)^{1/2} \tag{30}$$

$$\tan \phi = \frac{AD - BC}{AC + BD} \tag{31}$$

$$1 = 2\left(\frac{A^2 + B^2}{p^2 + q^2}\right)^{1/2} \tag{32}$$

where the coefficients α, β, γ, δ, A, B, C, D, p, and q, depend on the S'_{ij} and D'_{ij} coefficients and on r, ϕ, and ω.[6] Because the applied linearization technique fully considers the nonlinearities inherent in the machine, then the S'_{ij} and D'_{ij} coefficients are not constant, but depend on X_p by means of the a and b coefficients, as shown in Appendices A and B. This dependence represents the fundamental advantage of the proposed linearization procedure with respect to the traditional ones.[1] Therefore, Eqs. (28–32) represent a system of five nonlinear simultaneous algebraic equations in four unknowns, X_p, r, ω, and ϕ. This result implies that one equation, e.g., Eq. (32), is redundant and, thus, may be eliminated.

Bearing in mind Eqs. (3), (4), (25), and (26), the relative laws of motion of the power piston and displacer, namely, y_p and y_d, are

$$y_p(t) = (Y_p/2)\sin[(\omega t - \phi_{yd}) - \theta] \tag{33}$$

$$y_d(t) = (\sigma Y_p/2)\sin(\omega t - \phi_{yd}) \tag{34}$$

where Y_p, σ, ϕ_{yd}, and θ are given in Appendix C.

The cyclic average thermodynamic power $\bar{\dot{W}}_t$, i.e., the average power delivered by the lossless machine, is

$$\bar{\dot{W}}_t = \bar{\dot{W}}_e - \bar{\dot{W}}_c \tag{35}$$

The cyclic average powers, $\bar{\dot{W}}_e$ and $\bar{\dot{W}}_c$, delivered by the expansion and compression spaces, are

$$\bar{\dot{W}}_e = f\oint p_{w,l}\,\mathrm{d}V_e, \qquad \bar{\dot{W}}_c = -f\oint p_{w,l}\,\mathrm{d}V_c \tag{36}$$

where $p_{w,l}$ is the linearized cycle gas pressure given by Eq. (8), and the overbar sign (-) allows $\bar{\dot{W}}_c$ to be taken positive. The power $\bar{\dot{W}}_t$ may be also calculated as follows:

$$\bar{\dot{W}}_t = \bar{\dot{W}}_u + \bar{\dot{W}}_{\mathrm{dis}}$$

The cyclic average useful power $\bar{\dot{W}}_u$ delivered by the machine at the engine/load device interface is

$$\bar{\dot{W}}_u = -f\oint F_{\mathrm{ld-1},l}\,\mathrm{d}y_p \tag{37}$$

where $F_{\mathrm{ld-1},l}$ is the linearized force exerted by the LD–L subsystem on the piston motion, given by Eq. (19). The $\bar{\dot{W}}_u$ power is less than the $\bar{\dot{W}}_t$ power because of the dissipated power $\bar{\dot{W}}_{\mathrm{dis}}$, which accounts for the gas spring and bounce space hysteresis losses and for the heat exchangers and regenerator flow losses. From the equations of motion [Eqs. (1) and (2)], it follows that

$$\bar{\dot{W}}_{\mathrm{dis}} = f\oint\left(\Delta p_{w,l}\frac{A_p}{2} + D_{b,H}\dot{y}_p\right)\mathrm{d}y_p + f\oint\left[D_{\mathrm{gs},H}\dot{y}_d - \Delta p_{w,l}\left(A_d - \frac{A_r}{2}\right)\right]\mathrm{d}y_d \tag{38}$$

where $\Delta p_{w,l}$ is the linearized pressure drop given by Eq. (12). In the case considered up to now of an FPSE/LD–L system modeled by a linearization procedure, the relative displacer and piston motions are sinusoidal when the machine is running at a periodic steady state, as shown by Eqs. (33) and (34). Therefore, the cyclic integration of the two relationships defined by Eq. (36) gives

$$\bar{\dot{W}}_e = -\frac{1}{2}\,\omega\left(\frac{Y_p}{2}\right)^2\sigma\, b_w \sin\theta\, A_d\left(\frac{\partial p_w}{\partial y_p}\right)_{0,0} \tag{39}$$

$$\bar{\dot{W}}_c = -\frac{1}{2}\,\omega\left(\frac{Y_p}{2}\right)^2\sigma\, b_w \sin\theta\left[(A_d - A_r)\left(\frac{\partial p_w}{\partial y_p}\right)_{0,0} + A_p\left(\frac{\partial p_w}{\partial y_d}\right)_{0,0}\right] \tag{40}$$

The partial derivatives of p_w calculated in $(y_p, y_d) = (0, 0)$ depend on the thermodynamic model, e.g., either isothermal or adiabatic,[7] selected for the working gas circuit of the engine. In addition, the cyclic integration of Eq. (37), where the relative piston displacement is sinusoidal, gives

$$\bar{\dot{W}}_u = \tfrac{1}{2} b_{\mathrm{ld-,1D}} D_{\mathrm{ld-1}}(\omega Y_p/2)^2 \tag{41}$$

as well as the cyclic integration of Eq. (38), where the relative piston and displacer motions are sinusoidal and the forces exerted by the linearized pressure drop are given by Eqs. (13) and (14), gives

$$\bar{\dot{W}}_{\mathrm{dis}} = (\tfrac{1}{2})[(D_{fldp} - D_{flp} + D_{b,H}) - 2\sigma\cos\theta\, D_{fldp} + \sigma^2(D_{fldp} + D_{fld} + D_{\mathrm{gs},H})](\omega Y_p/2)^2 \tag{42}$$

Complete FPSE/LD–L Model and Its Applications

Equations (28–31) and (41) represent the dynamic behavior of any FPSE/LD–L system, whereas the influence of a particular LD–L subsystem as well as the thermodynamic behavior of a specific FPSE are described by means of the S'_{ij} and D'_{ij} coefficients, which are defined in Appendix B. Their algebraic expressions depend on the adopted load device and on the thermodynamic model, e.g., either isothermal or adiabatic,[7] selected for the working gas circuit of the machine. It may be noted that the adiabatic thermodynamic model is much closer to the actual behavior of the cycle fluid circuit of the machine than the isothermal one.[7]

Complete expressions of the adiabatic $D'_{ij,w}$ and $S'_{ij,w}$ coefficients are given in Ref. 10. The adiabatic damping coefficients depend, among other machine parameters, on the working gas temperatures in the heater and cooler, $T_{w,h}$ and $T_{w,k}$. The adiabatic stiffness coefficients depend also on the cyclic average temperatures in the expansion and compression spaces, $\bar{T}_e$ and $\bar{T}_c$, which are functions of the piston stroke X_p.[10]

The isothermal $D'_{ij,w}$ and $S'_{ij,w}$ coefficients, instead, are given in Refs. 3 and 7, and depend only on $T_{w,h}$ and $T_{w,k}$. In fact, in the case of isothermal behavior of the working spaces, we have $\bar{T}_e = T_{w,h}$, and $\bar{T}_c = T_{w,k}$.

However, in the isothermal and adiabatic models[7] the heater and cooler walls are maintained isothermally at the inlet temperatures of the heating (hot source) and cooling (cold sink) outside fluids, $T_{\mathrm{hf},i}$ and $T_{kf,i}$, and the heat exchanger cycle gas temperatures, $T_{w,h}$ and $T_{w,k}$, are equal to their associated wall temperatures. In addition, the regenerator effectiveness is considered unitary. Therefore, the heat exchangers (including the regenerator) are perfectly effective ($\varepsilon_h = \varepsilon_k = \varepsilon_{\mathrm{rg}} = 1$) in the isothermal and adiabatic models.

This limitation may be removed applying the closed-form thermal analysis developed by de Monte,[11] which allows the heat transfer processes at the FPSE heater, cooler, and regenerator, described by the following equations:

$$\varepsilon_h C_{hf}(T_{\mathrm{hf},i} - T_{w,h}) = \bar{\dot{W}}_e + \hat{\dot{Q}}_{\mathrm{rg}}(1 - \varepsilon_{\mathrm{rg}}) = \bar{\dot{Q}}_h \tag{43}$$

$$\varepsilon_k C_{kf}(T_{w,k} - T_{kf,i}) = \bar{\dot{W}}_c + \hat{\dot{Q}}_{\mathrm{rg}}(1 - \varepsilon_{\mathrm{rg}}) = \bar{\dot{Q}}_k \tag{44}$$

to be coupled to the FPSE dynamic (including the load device) and thermodynamic behavior, described by Eqs. (28–31) and (41).

In Eqs. (43) and (44), the cyclic average powers $\bar{\dot{W}}_e$ and $\bar{\dot{W}}_c$ are given by Eqs. (39) and (40), respectively; the heat-exchange effectiveness, ε_h, ε_k, and $\varepsilon_{\mathrm{rg}}$ may be accurately calculated as shown in Ref. 11, and the average heat transfer rate $\hat{\dot{Q}}_{\mathrm{rg}}$, transferred unidirectionally from the working gas to the regenerator matrix in the case of ideal behavior of the regenerator, is given in Ref. 12 and depends on the piston stroke.

Therefore, even if the nonlinearities inherent in the machine are fully neglected ($a = 1$, $b = 1$), the imperfect heat exchange of the heater, cooler, and regenerator implies that the D'_{ij} and S'_{ij} coefficients (both adiabatic and isothermal) are piston stroke dependent. In fact, the temperatures $T_{w,h}$ and $T_{w,k}$ appearing in these coefficients depend on X_p by means of the heat transfer equations [Eqs. (43) and (44)]. It may be noted that the adiabatic S'_{ij} coefficients also depend on X_p by means of $\bar{T}_e$ and $\bar{T}_c$.

Thus, the complete FPSE/LD–L model is obtained simply by linking Eqs. (28–31) and (41) to Eqs. (43) and (44). This model may be used for the performance prediction and the design of the machine. However, there are important differences in the obtained results according to the assumptions made, as shown in the following sections.

Neglected Nonlinearities

The following hypotheses are made: 1) The nonlinearities are neglected, 2) the working spaces are isothermal, and 3) the heat exchangers are perfectly effective. In the validity of these hypotheses, the a and b terms are equal to 1, and the temperatures $\bar{T}_e$ and $\bar{T}_c$ are constant and equal, respectively, to $T_{w,h}$ and $T_{w,k}$. In addition, because of the ideal heat exchangers, Eqs. (43) and (44) do not have any physical meaning, and the FPSE/LD–L model is reduced to Eqs. (28–31) and (41), with $T_{w,h} = T_{\mathrm{hf},i}$ and $T_{w,k} = T_{kf,i}$. Therefore, the D'_{ij} and S'_{ij} coefficients are independent of X_p.

Now, if the model is used for the performance prediction of the machine, the user may calculate only r, ω, and ϕ, but cannot absolutely estimate the piston stroke X_p and, therefore, the powers $\bar{\dot{W}}_u$, $\bar{\dot{W}}_t$, and $\bar{\dot{W}}_{\mathrm{dis}}$. In fact, we have four equations [Eqs. (28–31)], in three unknown variables, r, ω, and ϕ. The piston stroke X_p does not appear.

On the other hand, if the model is used for design purposes, then once $\bar{\dot{W}}_u$ and ω are assigned, and some other data are fixed on the basis of experience, it is possible to obtain the remaining variables (five) by solving Eqs. (28–31) and (41). The restriction [Eq. (27)] must also be verified.[6] In particular, it is possible to obtain the design value of the load for cyclic steady operation of the machine as well as the design values of the other operating parameters. These values are critical because a free-piston Stirling machine modeled according to the simplifying assumptions listed earlier is unstable; that is, it has steady oscillations for just one value of the operational parameters, e.g., the load. In other words, once the machine is designed, any variation of the operational parameters will result in either internal collisions or engine stopping. Obviously, this result does not agree with the experience.

Therefore, when designing a machine it is possible to ensure a periodic steady operation, but not a quite stable operation.

Included Nonlinearities

If the applied dynamic linearization technique fully considers the nonlinearities and an adiabatic behavior for the FPSE working spaces is assumed, as done by the authors in this paper, then the D'_{ij} and S'_{ij} coefficients are not constant, but depend on X_p by means of the a and b coefficients (see Appendices A and B) and by means of the $\bar{T}_e$ and $\bar{T}_c$ temperatures.[10] In addition, because the heat exchangers are not perfectly effective, the D'_{ij} and S'_{ij} coefficients also depend on X_p by means of the temperatures $T_{w,h}$ and $T_{w,k}$ appearing in the heat transfer equations [Eqs. (43) and (44)].

Now, if the model is used for the performance prediction of the engine, the six unknown variables X_p, r, ω, ϕ, $T_{w,h}$, and $T_{w,k}$ may be obtained by solving Eqs. (28–31) and (43) and (44), which represent a system of six nonlinear simultaneous algebraic equations. Then, Eqs. (39–42) allow the powers $\bar{\dot{W}}_e$, $\bar{\dot{W}}_e$ ($\Rightarrow \bar{\dot{W}}_t$), $\bar{\dot{W}}_u$, and $\bar{\dot{W}}_{\mathrm{dis}}$, to be calculated.

On the other hand, if the model is used for the design of the engine, once $\bar{\dot{W}}_u$ and ω are assigned and some other data are fixed on the basis of experience, it is possible to obtain the remaining variables (seven) by solving Eqs. (28–31), (41), (43), and (44). The restriction [Eq. (27)] must be verified. In particular, it is possible to obtain the design values of the operating conditions, e.g., the load, for cyclic steady operation of the machine. These values are not critical, because of 1) the presence of nonlinearities, 2) the nonisothermal behavior of the working spaces, and 3) the imperfect effectiveness of the heat exchangers (including the regenerator). In other words, once the machine is designed, any variation of the operational parameters, in particular of the load, will not result in general in either internal collisions or engine stopping.

This means that a free-piston Stirling machine, described by the present improved model, is an inherently stable machine as confirmed by experimental results.

Stable Operation

We have six equations, (28–31), (43), and (44), and six unknown variables, r, ω, ϕ, X_p, $T_{w,k}$, and $T_{w,k}$, so that every value of the load corresponds to a set of values for X_p, r, ω, ϕ, $T_{w,h}$, and $T_{w,k}$. In other words, if the load increases (or decreases), the oscillations of the piston and displacer will be convergent (or divergent) until X_p, r, ω, ϕ, $T_{w,h}$, and $T_{w,k}$ will not reach a new set of values able to satisfy Eqs. (28–31), (43), and (44) and, in particular the oscillation criterion, expressed by Eqs. (27) and (28), with the new value assumed by the load. Obviously, the new set of values of X_p, r, ω, ϕ, $T_{w,h}$, and $T_{w,k}$, which defines a new periodic steady state for the machine, must be consistent with its geometric constraints, to avoid internal collisions between the engine parts. It is easy to verify that the damping D'_{ij} and stiffness S'_{ij} coefficients of the machine increase with the piston stroke. Therefore, at large piston strokes, the damping and spring constants are increased, reducing the piston and displacer strokes and, consequently, the possibility of internal collisions.

Thus, the elements that have a stabilizing effect on the operation of an FPSE connected to a generic LD–L subsystem are 1) the nonlinearities associated with p_w, p_{gs}, and p_b; 2) the nonlinearity associated with Δp_w; 3) the nonlinear loads [see Eq. (18)]; 4) the nonisothermal behavior of the expansion and compression spaces; and 5) the heat transfer through a finite temperature difference between the cycle gas and the outside fluids (heating and cooling fluids).

Because the proposed FPSE/LD–L model allows the algebraic expressions of the various stabilizing elements to be determined [the coefficients a and b are given in Appendix A, the temperatures $\bar{T}_e$ and $\bar{T}_c$ in Ref. 10, and the temperatures $T_{w,h}$ and $T_{w,k}$ are defined by means of Eqs. (43) and (44)], this method can be used to ensure the machine not only a periodic steady operation, but most of all a quite stable operation.

The inherent stability is a very important step for the FPSE design, but only in the initial stages. In other words, it may not be considered sufficient to make a free-piston machine work properly. In fact, the FPSE/LD–L systems will, in general, have a load-dependent operating frequency, as well as a load-dependent piston stroke (Fig. 1), whereas there are many applications (FPSE/LA–sL systems) where an almost constant voltage output, i.e., constant frequency and constant piston amplitude, is demanded for a wide range of loads.

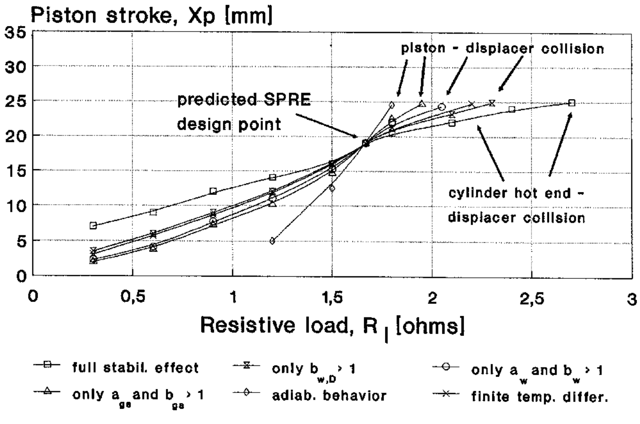

Fig. 1 Effect of the various stabilizing elements on SPRE operation.

Therefore, typical free-piston engine external power controls,[5,13] as well as internal power controls recently proposed by Lane and Beale,[14] must be used. Both the internal and external power controls, matching suitably generated power to load requirements, actually allow the piston stroke and frequency to be held almost constant during engine operation. However, an inherently constant frequency may be obtained also in a different way.[8]

The external power control reduces the machine efficiency over a substantial range of output power, increases complexity, and is not reliable and cost-effective. Instead, the internal power control, which is a variable spring between the piston and displacer actuated by a combination of a mechanical and pneumatic feedback loop, is an efficient, cost-effective, reliable, and fail-safe control mechanism. It artificially induces nonlinearities, limiting the piston oscillation amplitude.

Effect of Stabilizing Elements on SPRE Operation

In the preceding section we have singled out all of the elements that have a stabilizing effect on the operation of an FPSE connected to a generic LD–L subsystem.

To show the possible applications of the developed theory, we have considered the Space Power Research Engine (SPRE),[15,16] which is a 12.5-kWe single-cylinder FPSE with displacer sprung to ground connected to a linear alternator load, whose testing and results have been carried out by Cairelli et al.[17]

In this machine, the $b_{\mathrm{ld-1},S}$ and $b_{\mathrm{ld-1},D}$ terms are independent of X_p. In fact, the LA–sL subsystem connected to the machine produces a force resisting the power piston motion that depends linearly on y_p and $\dot{y}_p$,[8,9] that is $n_S = 1$ and $n_D = 1$ in Eq. (18) $\Rightarrow b_{\mathrm{ld-1},S} = 1$, $b_{\mathrm{ld-1},D} = 1$ (see Appendix A). Therefore, this force does not have any stabilizing effect on the machine operation. In addition, $a_b \simeq 1$ and $b_b \simeq 1$, because the SPRE bounce space is very large to contain the electromechanical device.

Figure 1, obtained assuming an adiabatic behavior for the working spaces, shows the effect on the SPRE operation of each stabilizing element considered separately when the R_l load electrical resistance changes. In particular, the nonlinearity associated with Δp_w (curve ⧖) is convenient in the SPRE because it improves its stability more than the other stabilizing elements, and in particular more than the nonlinearities associated with p_w (curve ○) and p_{gs} (curve △). Also the thermodynamic losses because of the heat transfer through a finite temperature difference between the cycle gas and the outside fluids (heating and cooling fluids) have a high stabilizing effect on the SPRE operation (curve ×). The adiabatic behavior of the working spaces, instead, has a very small stabilizing influence on the machine operation, as shown in Fig. 1 (curve ◇). This influence is even smaller because the behavior of the working spaces is not perfectly adiabatic. When all of the stabilizing elements inherent in the SPRE are involved together (curve □), as it actually happens while the machine is running, the load collision value is $R_l = 2.7\Omega$. However, the nonlinearity associated with Δp_w reduces the machine efficiency.

In fact, if the Ψ_{tr} coefficient appearing in Eqs. (9) and (12) is increased from its design value (3.6×10^7 Pa s^2/m^6, predicted by means of the developed theory), the stability of the machine will increase, as shown in Fig. 2a. On the other hand, the cyclic average power $\bar{\dot{W}}_{\mathrm{dis,tr}}$, dissipated because of the turbulent viscous flow losses in the heater and cooler will also increase, as shown in Fig. 2b. In particular, when $\Psi_{\mathrm{tr}} = 5.6 \times 10^7$ Pa s^2/m^6, the collision between the displacer and the cylinder hot-end is reached with $R_l = 3.4\Omega$ (Fig. 2a), but the dissipated power $\bar{\dot{W}}_{\mathrm{dis,tr}}$ is about 1850 W (Fig. 2b), reducing the SPRE efficiency. To avoid a low efficiency and to keep a high stability, external or internal load control mechanisms must be used.

Lane and Beale,[14] as well as Corey,[18] have proposed a new arrangement of free-piston Stirling machine with displacer sprung to ground, where the displacer is completely resonated by planar mechanical springs. In other words, the displacer uses metallic springs in place of pressure-dependent gas springs. This choice has the advantage of eliminating the centering port system for the displacer (large advantage) and the energy losses that are involved in the gas spring; that is, the hysteresis losses and the pressure losses caused by mass leakage through the clearance seal and centering port for displacer (this is small advantage, because these losses are quite small[19]). But it has the disadvantage of eliminating the stabilizing effect of the nonlinearities associated with p_{gs}, unless a nonlinear mechanical spring is used. As shown in Fig. 1, the p_{gs} nonlinearities are able to stabilize the SPRE machine operation when the resistive load R_l exceeds its design value (1.66Ω), provided it stays below the collision value (1.95Ω).

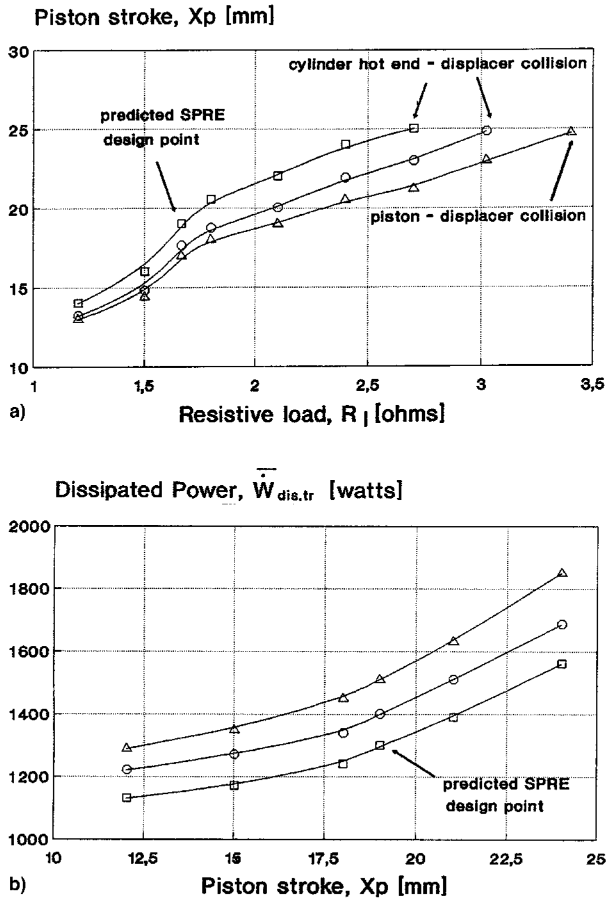

Fig. 2 Effect of the turbulent viscous flow losses on a) SPRE stability and b) dissipated power $\bar{\dot{W}}_{\mathrm{dis,tr}}$. □, $\Psi_{tr} = 3.6 \times 10^7$ Pa s^2/m^6; ○, $\Psi_{tr} = 4.6 \times 10^7$ Pa s^2/m^6; and △, $\Psi_{tr} = 5.6 \times 10^7$ Pa s^2/m^6.

Conclusions

The FPSE behavior is influenced by complex nonlinear thermo-fluid-dynamic phenomena that are generally neglected by the majority of linear dynamic analyses.

The inclusion of nonlinear thermo-fluid-dynamic terms by means of the presented methodology allows a better interpretation of the machine operation to be obtained, particularly when the operating conditions change.

The choice of design parameters allows the stabilizing elements inherent in the machine to be calibrated and, therefore, an FPSE that is unconditionally stable with the load to be designed. This is a very important step in the initial stages of the design because the FPSE configuration often has a significant bearing on this question. However, complex external or internal power-control mechanisms must always be used in free-piston machines because of two basic reasons:

1) Most FPSE/LA applications demand constant voltage output, that is, constant frequency and constant power piston stroke for a wide range of loads.

2) An FPSE designed for maximum efficiency will have laminar flow heat exchangers that make inherent stability almost impossible.

The Space Power Research Engine has been considered to illustrate the stability characteristics of a free-piston Stirling machine connected to a linear alternator load, when the power controls are not working.

Appendix A: Nonlinear Terms

The calculation of the a and b coefficients, which take account of the nonlinear thermo-fluid-dynamic terms inherent in the free-piston machine, requires the knowledge of the relative laws of motion of the piston and displacer. They may be assumed sinusoidal and, therefore, expressed by Eqs. (33) and (34). This assumption is justified on the basis of the linearization technique applied to the dynamic equations, which rigorously leads to sinusoidal laws of motion in the case of cyclic steady state of the machine.[6]

Working Gas Circuit: Pressure

By solving Eqs. (6) and (7), where the piston and displacer motions are sinusoidal oscillations, we obtain for $a_{w,m}$ (m is even) and $b_{w,m}$ (m is odd) the following expressions:

$$a_{w,m}p_w(0,0) = \frac{1}{m!}\left(\frac{Y_p}{2}\right)^m \sum_{k=0}^{m}\left[\left(\frac{\partial^m p_w}{\partial y_p^k \partial y_d^{m-k}}\right)_{0,0}\binom{m}{k}\sigma^{m-k} I_{m,k}\right] \tag{A1}$$

$$b_{w,m}\sum_{k=0}^{1}\left[\binom{1}{k}\left(\frac{\partial p_w}{\partial y_p^k \partial y_d^{1-k}}\right)_{0,0}\sigma^{1-k}(A_p I_{1,k,p} + A_r\sigma I_{1,k,d})\right] = \frac{1}{m!}\left(\frac{Y_p}{2}\right)^{m-1}\sum_{k=0}^{m}\left[\binom{m}{k}\left(\frac{\partial^m p_w}{\partial y_p^k \partial y_d^{m-k}}\right)_{0,0}\sigma^{m-k}J_{m,k}\right] \tag{A2}$$

where

$$I_{m,k} = \frac{1}{2\pi}\oint [\sin(\omega t - \theta)]^k[\sin(\omega t)]^{m-k}\,\mathrm{d}\omega t \tag{A3}$$

$$I_{1,k,p} = \oint [\sin(\omega t - \theta)]^k[\sin(\omega t)]^{1-k}\,\mathrm{d}[\sin(\omega t - \theta)]$$

$$I_{1,k,d} = \oint [\sin(\omega t - \theta)]^k[\sin(\omega t)]^{1-k}\,\mathrm{d}[\sin(\omega t)]$$

$$J_{m,k} = (A_p I_{m,k,p} + A_r\sigma I_{m,k,d})$$

$$I_{m,k,p} = \oint [\sin(\omega t - \theta)]^k[\sin(\omega t)]^{m-k}\,\mathrm{d}[\sin(\omega t - \theta)]$$

$$I_{m,k,d} = \oint [\sin(\omega t - \theta)]^k[\sin(\omega t)]^{m-k}\,\mathrm{d}[\sin(\omega t)]$$

The even- and odd-order partial derivatives of p_w calculated in $(y_p, y_d) = (0, 0)$, appearing in Eqs. (A1) and (A2), depend on the thermodynamic model, e.g., either isothermal or adiabatic,[7] selected for the working gas circuit of the machine. Both a_w and b_w depend on X_p, r, and ϕ, by means of Y_p, σ, and θ (see Appendix C).

Working Gas Circuit: Pressure Drop

The expression [Eq. (10)] for $\dot{V}_w$, where the relative piston and displacer motions are sinusoidal, becomes

$$\dot{V}_w(t) = \dot{V}_{w,\max}\sin(\omega t - \phi_V)$$

where ϕ_V is the phase angle of $\dot{V}_w(t)$ with respect to y_d given by Eq. (34), and $\dot{V}_{w,\max}$ is

$$\dot{V}_{w,\max} = (Y_p\omega/4)[A_p^2 - 2A_p(2A_d - A_r)\sigma\cos\theta + (2A_d - A_r)^2\sigma^2]^{1/2}$$

Once the time expression of $\dot{V}_w$ is known, it is possible to calculate the $b_{w,D}$ coefficient by Eq. (11):

$$b_{w,D} = [8/(3\pi)]\dot{V}_{w,\max}$$

Therefore, the $b_{w,D}$ coefficient depends on ω, X_p, r, and ϕ, by means of Y_p, σ, and θ (see Appendix C).

Gas Spring

By solving Eqs. (16) and (17), where the relative law of motion of the displacer is assumed sinusoidal, we obtain for $a_{gs,m}$ (m is even) and $b_{gs,m}$ (m is odd) the following expressions:

$$a_{gs,m}p_{gs}(0) = \frac{1}{m!}\left(\frac{\mathrm{d}^m p_{gs}}{\mathrm{d}y_d^m}\right)_0\left(\sigma\frac{Y_p}{2}\right)^m I_{gs,m}$$

$$b_{gs,m}\left(\frac{\mathrm{d}p_{gs}}{\mathrm{d}y_d}\right)_0 = \frac{2}{(m+1)!}\left(\frac{\mathrm{d}^m p_{gs}}{\mathrm{d}y_d^m}\right)_0\left(\sigma\frac{Y_p}{2}\right)^{m-1}$$

where $I_{gs,m}$ is equal to $I_{m,k}$ given by Eq. (A3) when $k = 0$. It may be noted that the calculation of $b_{gs,m}$ does not require the knowledge of the displacer motion. The even- and odd-order derivatives of p_{gs} calculated in $y_d = 0$ can be easily evaluated if an ideal adiabatic behavior for the gas spring is assumed, as is usually done in the analyses of FPSEs.[3,7] Both a_{gs} and b_{gs} depend on $rX_p = X_d$ by means of $\sigma Y_p = Y_d$ (see Appendix C).

Bounce Space

The terms $a_{b,m}$ (m is even) and $b_{b,m}$ (m is odd) have the following expressions:

$$a_{b,m}p_b(0) = \frac{1}{m!}\left(\frac{\mathrm{d}^m p_b}{\mathrm{d}y_p^m}\right)_0\left(\frac{Y_p}{2}\right)^m I_{b,m}$$

$$b_{b,m}\left(\frac{\mathrm{d}p_b}{\mathrm{d}y_p}\right)_0 = \frac{2}{(m+1)!}\left(\frac{\mathrm{d}^m p_b}{\mathrm{d}y_p^m}\right)_0\left(\frac{Y_p}{2}\right)^{m-1}$$

where $I_{b,m}$ is equal to $I_{m,k}$ given by Eq. (A3) when $k = m$. The even- and odd-order derivatives of p_b calculated in $y_p = 0$ can be easily evaluated if an ideal adiabatic behavior for the bounce space is assumed, as previously stated for the gas spring. Both a_b and b_b depend on X_p by means of Y_p (see Appendix C).

If the FPSE is connected to a linear alternator, the bounce space will be very large to contain the electromechanical device. In this case the thermodynamic behavior of the bounce space is better described by means of an ideal isothermal model, and the nonlinearities associated with the pressure p_b are negligible: $a_b \simeq 1$, $b_b \simeq 1$.

LD–L Subsystem

By solving Eq. (22), we get the $b_{ld-1,S}$ coefficient related to the stiffness of the LD–L subsystem:

$$b_{ld-1,S} = \frac{2}{n_S + 1}\left(\frac{Y_p}{2}\right)^{n_S - 1}$$

Assuming the relative law of motion of the power piston to be sinusoidal, from Eq. (21) we get the $b_{ld-1,D}$ coefficient related to the damping of the LD–L subsystem:

$$b_{ld-1,D} = 2[\omega(Y_p/2)]^{n_D - 1}\Pi_{ld-1}(n_D)$$

$$\Pi_{ld-1}(n_D) = (1/\pi)\beta_F(n_D/2 + 1, 0.5)$$

The symbol β_F indicates the beta function. The $b_{\text{ld}-1,S}$ coefficients depends on X_p by means of Y_p (see Appendix C), whereas $b_{\text{ld}-1,D}$ also depends on ω.

In most FPSE applications the contribution of the nonlinear terms associated with the pressures p_w, p_{gs}, and p_b, is small and decreases quickly with the order of derivation.[7] Therefore, in many cases it is possible, with good approximation, to consider only the contribution of the nonlinear terms of second- and third-order.[20]

Appendix B: FPSE/LD–L Stiffness and Damping Coefficients

The S'_{ij} and D'_{ij} coefficients of an FPSE/LD–L system, which appear in Eqs. (23) and (24), may be decomposed into the sum of four terms as follows:

$$S'_{ij} = S'_{ij,w} + S'_{ij,\text{gs}} + S'_{ij,b} + S'_{ij,\text{ld}-1}$$

$$D'_{ij} = D'_{ij,w} + D'_{ij,\text{gs}} + D'_{ij,b} + D'_{ij,\text{ld}-1}$$

where $i = p$, d, and $j = p$, d. The right-hand coefficients are next given separately.

Working Gas Circuit

1) Stiffness coefficients per unit of moving element mass because of p_w

$$S'_{pj,w} = -b_w \frac{A_p}{M_p}\left[\left(\frac{\partial p_w}{\partial y_j}\right)_{0,0}(1 + \mu_{jc}) + \left(\frac{\partial p_w}{\partial y_k}\right)_{0,0}\mu_{jc}\right]$$

$$S'_{dj,w} = S'_{pj,w}\frac{A_r}{A_p}\frac{M_p}{M_d}, \qquad j = p, d$$

where $k = d$ when $j = p$, and $k = p$ when $j = d$. The expressions of the $S'_{ij,w}$ coefficients depend on the selected thermodynamic model, e.g., either isothermal or adiabatic,[7] which allows the odd-order partials of p_w in $(y_p, y_d) = (0, 0)$ to be evaluated.

2) Damping coefficients per unit of moving element mass because of Δp_w

$$D'_{pp,w} = \frac{D_{fldp} - D_{flp}(1 + \mu_{\text{pc}})}{M_p}$$

$$D'_{pd,w} = -\frac{D_{fldp} + D_{flp}\mu_{\text{dc}}}{M_p}$$

$$D'_{dp,w} = -\frac{D_{fldp} - D_{fld}\mu_{\text{pc}}}{M_d}$$

$$D'_{dd,w} = \frac{D_{fldp} + D_{fld}(1 + \mu_{\text{dc}})}{M_d}$$

where

$$D_{fldp} = \tfrac{1}{2}\Psi_{w,l}A_p(A_d - A_r/2)$$

$$D_{flp} = \tfrac{1}{4}\Psi_{w,l}A_p[(2A_d - A_r) - A_p]$$

$$D_{fld} = \tfrac{1}{2}\Psi_{w,l}[(2A_d - A_r) - A_p](A_d - A_r/2)$$

The $\Psi_{w,l}$ coefficient is defined by Eq. (12), and depends on X_p, r, ϕ, and ω, by means of $b_{w,D}$ given in Appendix A.

The preceding definitions of the $S'_{ij,w}$ and $D'_{ij,w}$ coefficients are valid for any beta FPSE, with the exception of the thermomechanical generator.[9]

Gas Spring

1) Stiffness coefficients per unit of moving element mass because of p_{gs}

$$S'_{pp,\text{gs}} = S'_{pd,\text{gs}} = 0$$

$$S'_{dp,\text{gs}} = b_{\text{gs}}\frac{A_r}{M_d}\mu_{\text{pc}}\left(\frac{\text{d}p_{\text{gs}}}{\text{d}y_d}\right)_0$$

$$S'_{dd,\text{gs}} = b_{\text{gs}}\frac{A_r}{M_d}(1 + \mu_{\text{dc}})\left(\frac{\text{d}p_{\text{gs}}}{\text{d}y_d}\right)_0$$

2) Damping coefficients per unit of moving element mass because of gas spring hysteresis losses

$$D'_{pp,\text{gs}} = D'_{pd,\text{gs}} = 0, \quad D'_{dp,\text{gs}} = (D_{\text{gs},H}/M_d)\mu_{\text{pc}}$$

$$D'_{dd,\text{gs}} = (D_{\text{gs},H}/M_d)(1 + \mu_{\text{dc}})$$

where $D_{\text{gs},H}$ may be calculated as indicated in Ref. 7.

The previous definitions of the $S'_{ij,\text{gs}}$ and $D'_{ij,\text{gs}}$ coefficients are valid only for beta FPSEs with displacer sprung to ground.

Bounce Space

1) Stiffness coefficients per unit of moving element mass because of p_b

$$S'_{pp,b} = b_b\frac{A_p}{M_p}(1 + \mu_{\text{pc}})\left(\frac{\text{d}p_b}{\text{d}y_p}\right)_0$$

$$S'_{pd,b} = b_b\frac{A_p}{M_p}\mu_{\text{dc}}\left(\frac{\text{d}p_b}{\text{d}y_p}\right)_0$$

$$S'_{dp,b} = S'_{dd,b} = 0$$

2) Damping coefficients per unit of moving element mass because of bounce space hysteresis losses

$$D'_{pp,b} = (D_{b,H}/M_p)(1 + \mu_{\text{pc}})$$

$$D'_{pd,b} = (D_{b,H}/M_p)\mu_{\text{dc}}, \qquad D'_{dp,b} = D'_{dd,b} = 0$$

where $D_{b,H}$ may be calculated as indicated in Ref. 7. If the bounce space is very large, the hysteresis losses may be neglected.

The previous definitions of the $S'_{ij,b}$ and $D'_{ij,b}$ coefficients are valid for beta FPSEs with displacer sprung to ground and with displacer sprung to piston.

LD–L Subsystem

1) Stiffness coefficients per unit of moving element mass

$$S'_{pp,\text{ld}-1} = [(b_{\text{ld}-1,S}S_{\text{ld}-1})/M_p](1 + \mu_{\text{pc}})$$

$$S'_{pd,\text{ld}-1} = [(b_{\text{ld}-1,S}S_{\text{ld}-1})/M_p]\mu_{\text{dc}}$$

$$S'_{dp,\text{ld}-1} = S'_{dd,\text{ld}-1} = 0$$

2) Damping coefficients per unit of moving element mass

$$D'_{pp,\text{ld}-1} = [(b_{\text{ld}-1,D}D_{\text{ld}-1})/M_p](1 + \mu_{\text{pc}})$$

$$D'_{pd,\text{ld}-1} = [(b_{\text{ld}-1,D}D_{\text{ld}-1})/M_p]\mu_{\text{dc}}$$

$$D'_{dp,\text{ld}-1} = D'_{dd,\text{ld}-1} = 0$$

where coefficients $b_{\text{ld}-1,S}$ and $b_{\text{ld}-1,D}$ are given in Appendix A.

Appendix C: Relative Piston and Displacer Motions

The dynamic quantities Y_p, σ, ϕ_{yd}, and θ, characterizing the relative laws of motion [Eqs. (33) and (34)], are

$$Y_p = X_p\mathcal{F}_p, \qquad \sigma = r(\mathcal{F}_d/\mathcal{F}_p)$$

$$\tan \phi_{yd} = (\tan \phi)\mathscr{F}_{\phi d}, \qquad \tan \theta = (\tan \phi)\mathscr{F}_{\phi pd}$$

where $\mathscr{F}_p$, $\mathscr{F}_d$, $\mathscr{F}_{\phi d}$, and $\mathscr{F}_{\phi pd}$ are given by

$$\mathscr{F}_p = [(1 + \mu_{pc})^2 + 2(1 + \mu_{pc})\mu_{dc} r \cos \phi + (\mu_{dc} r)^2]^{1/2}$$

$$\mathscr{F}_d = [(1 + \mu_{dc})^2 + 2(1 + \mu_{dc})\mu_{pc} \cos \phi / r + (\mu_{pc}/r)^2]^{1/2}$$

$$\mathscr{F}_{\phi d} = \frac{\mu_{pc} \cos \phi}{(1 + \mu_{dc})r + \mu_{pc} \cos \phi}$$

$$\mathscr{F}_{\phi pd} = \frac{r \cos \phi[(1 + \mu_{pc})(1 + \mu_{dc}) - \mu_{pc}\mu_{dc}]}{r \cos \phi[(1 + \mu_{pc})(1 + \mu_{dc}) + \mu_{pc}\mu_{dc}] + \chi}$$

$$\chi = (1 + \mu_{pc})\mu_{pc} + r^2(1 + \mu_{dc})\mu_{dc}$$

The $\mathscr{F}_p$, $\mathscr{F}_d$, and $\mathscr{F}_{\phi pd}$ coefficients are greater than 1 ($\mathscr{F}_{\phi d}$ greater than 0), although they are approximately equal to 1 ($\mathscr{F}_{\phi d}$ to 0) because in most FPSE applications $M_c >> M_p$ and $M_c >> M_d$, such that μ_{pc} and μ_{dc} approach 0.

References

[1]Kankam, M. D., and Rauch, J. S., "Comparative Survey of Dynamic Analyses of Free-Piston Stirling Engines," *Proceedings of the 26th Intersociety Energy Conversion Engineering Conference* (Boston, MA), Vol. 5, American Nuclear Society, La Grange Park, IL, 1991, pp. 314–319.

[2]Chen, N. C. J., and Griffin, F. P., "Linear Harmonic Analysis of Free-Piston Stirling Engines," Martin Marietta Energy Systems, Inc., Oak Ridge Natl. Lab., ORNL/con 172, Oak Ridge, TN, 1986.

[3]Benvenuto, G., de Monte, F., and Farina, F., "Dynamic Behavior Prediction of Free-Piston Stirling Engines," *Proceedings of the 25th Intersociety Energy Conversion Engineering Conference* (Reno, NV), Vol. 5, AIChE, 1990, pp. 346–351.

[4]Ulusoy, N., and McCaughan, F., "Nonlinear Analysis of Free-Piston Stirling Engines," *Proceedings of the 28th Intersociety Energy Conversion Engineering Conference* (Atlanta, GA), Vol. 2, American Chemical Society, Washington, DC, 1993, pp. 739–744.

[5]Berchowitz, D. M., "Operational Characteristics of Free-Piston Stirling Engines," *Proceedings of the 23rd Intersociety Energy Conversion Engineering Conference* (Denver, CO), American Society of Mechanical Engineers, New York, 1988, pp. 107–112.

[6]de Monte, F., and Benvenuto, G., "Reflections on Free-Piston Stirling Engines, Part 1: Cyclic Steady Operation," *Journal of Propulsion and Power*, Vol. 14, No. 4, 1998, pp. 499–508.

[7]Urieli, I., and Berchowitz, D. M., *Stirling Cycle Engine Analysis*, Adam Hilger Ltd., Bristol, England, UK, 1984.

[8]Benvenuto, G., and de Monte, F., "Analysis of Free-Piston Stirling Engine/Linear Alternator Systems Part 1: Theory," *Journal of Propulsion and Power*, Vol. 11, No. 5, 1995, pp. 1036–1046.

[9]Benvenuto, G., and de Monte, F., "Analysis of Free-Piston Stirling Engine/Linear Alternator Systems Part 2: Results," *Journal of Propulsion and Power*, Vol. 11, No. 5, 1995, pp. 1047–1055.

[10]Benvenuto, G., and de Monte, F., "An Adiabatic Approach for the Design and the Response-to-Load-Evaluation of Free-Piston Stirling Engines," *Proceedings of the 27th Intersociety Energy Conversion Engineering Conference* (San Diego, CA), Vol. 5, Society of Automotive Engineers, Warrendale, PA, 1992, pp. 23–31.

[11]de Monte, F., "Thermal Analysis of the Heat Exchangers and Regenerator in Stirling Cycle Machines," *Journal of Propulsion and Power*, Vol. 13, No. 3, 1997, pp. 404–411.

[12]Benvenuto, G., and de Monte, F., "A Simple Approach to Calculate the Regenerator Effectiveness in Stirling Machines," *Proceedings of the 7th International Conference on Stirling Cycle Machines*, Japanese Society of Mechanical Engineers, Tokyo, 1995, pp. 307–313.

[13]Rauch, J. S., Kankam, M. D., Santiago, W., and Madi, F. J., "A Free-Piston Stirling Engine/Linear Alternator Controls and Load Interaction Test Facility," *Proceedings of the 27th Intersociety Energy Conversion Engineering Conference* (San Diego, CA), Vol. 5, Society of Automotive Engineers, Warrendale, PA, 1992, pp. 309–314.

[14]Lane, N. W., and Beale, W. T., "Free-Piston Stirling Design Features," *Proceedings of the 8th International Stirling Engine Conference and Exhibition*, C. M. Bartolini & F. Caresana, Ancona, Italy, 1997, pp. 267–276.

[15]Brown, A. T., "Space Power Demonstrator Engine, Phase I Final Report," NASA CR-179555, May 1987.

[16]Geng, S. M., and Tew, R. C., "Comparison of GLIMPS and HFAST Stirling Engine Code Predictions with Experimental Data," *Proceedings of the 27th Intersociety Energy Conversion Engineering Conference* (San Diego, CA), Vol. 5, Society of Automotive Engineers, Warrendale, PA, 1992, pp. 53–58.

[17]Cairelli, J. E., Geng, S. M., Wong, W. A., Swec, D. M., Doeberling, T. J., and Madi, F. J., "1993 Update on Results of SPRE Testing at NASA Lewis," *Proceedings of the 28th Intersociety Energy Conversion Engineering Conference* (Atlanta, GA), Vol. 2, American Chemical Society, Washington, DC, 1993, pp. 633–638.

[18]Corey, J. A., "Twinstar Free-Piston Stirling Engine: Description and Development Results for Prototype and Pre-Production Units," *Proceedings of the 7th International Conference on Stirling Cycle Machines*, Japanese Society of Mechanical Engineers, Tokyo, 1995, pp. 203–209.

[19]Benvenuto, G., de Monte, F., and Galli, G., "The Influence of Thermodynamic Losses on Free-Piston Stirling Engine Performance," *Proceedings of the 30th Intersociety Energy Conversion Engineering Conference* (Orlando, FL), Vol. 3, American Society of Mechanical Engineers, New York, 1995, pp. 443–450.

[20]Benvenuto, G., and de Monte, F., "The Effect of Nonlinear Thermo-Fluid-Dynamic Terms on Free-Piston Stirling Machine Stability," *Proceedings of the 31st Intersociety Energy Conversion Engineering Conference* (Washington, DC), Vol. 2, Inst. of Electrical and Electronics Engineers, New York, 1996, pp. 1224–1231.